

Mass Spectra in the Doubly Symmetric Theory of Infinite-Component Fields

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Abstract

We consider the problem of the characteristics of mass spectra in the doubly symmetric theory of fields transforming under the proper Lorentz group representations decomposable into an infinite direct sum of finite-dimensional irreducible representations. We show that there exists a range of free parameters of the theory where the mass spectra of fermions are quite satisfactory from the physical standpoint and correspond to the picture expected in the parton model of hadrons.

1. Introduction

In [1], [2], the beginning was laid for investigating the relativistically invariant theory of fields transforming under representations of the proper Lorentz group L_+^\uparrow that are decomposable into an infinite direct sum of finite-dimensional irreducible representations (we say that such fields belong to the ISFIR class).

The structure of relativistically invariant Lagrangians of any free fields was established and described by Gelfand and Yaglom [3], [4]; such Lagrangians have the form

$$\mathcal{L}_0 = \frac{i}{2}[(\Psi, \Gamma^\mu \partial_\mu \Psi) - (\partial_\mu \Psi, \Gamma^\mu \Psi)] - (\Psi, R\Psi). \quad (1)$$

For fields of the ISFIR class, the matrix operators Γ^μ involved in Lagrangian (1) contain an infinite number of arbitrary constants. Because of this, restriction to the theory of the ISFIR-class fields was performed in [1]; this restriction was achieved by imposing the condition that Lagrangian (1) is also invariant under the secondary-symmetry transformations¹⁾

$$\Psi'(x) = \exp(-iD^\mu \theta_\mu) \Psi(x). \quad (2)$$

In formula (2), the parameters θ_μ are polar or axial four-vectors of the orthochronous Lorentz group L_+^\uparrow , and the operators D^μ have a matrix realization. The chosen requirement leads to selecting a countable set of versions of the theory, each of which is characterized by a definite representation of the L_+^\uparrow group, by single-valued (up to a normalization constant) four-vector operators Γ^μ and D^μ , and by the operator R being proportional to the unit operator E .

To avoid infinite degeneration with respect to spin in the mass spectrum for the Lorentz group extension performed in [1] (such a degeneration is to be expected in accordance with the Coleman–Mandula theorem [7]), spontaneous breaking of the secondary symmetry was postulated in [2]: it was assumed that scalar components (with respect to the L_+^\uparrow group) of one or several bosonic infinite-component fields of the ISFIR class have nonzero vacuum expectations λ^i (the index i can, e.g., denote a chosen component of some representation of

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¹⁾The notion of double symmetry, consisting of the primary and secondary symmetries, was introduced in [5]. It can be considered a generalization of supersymmetry and the symmetry of the Gell-Mann–Levi σ -model [6].

the internal symmetry group $SU(3)$). Assuming that the operator R specifying the mass term in the Lagrangian of the free fermion field is entirely caused by spontaneous breaking of the secondary symmetry, we then have

$$R = \sum_i \lambda^i Q_i^{(0,1)00}. \quad (3)$$

The operators $Q_i^{(0,1)00}$ occurring in relation (3) originate from the Lagrangian of the fermion-boson interaction

$$\mathcal{L}_{\text{int}} = \sum_{i', \tau, l, m} (\psi(x), Q_{i'}^{\tau lm} \varphi_{\tau lm}^{i'}(x) \psi(x)), \quad (4)$$

where $\psi(x)$ is the fermionic field, $\varphi_{\tau lm}^{i'}(x)$ is the component of the bosonic field characterized by the finite-dimensional irreducible representation $\tau = (l_0, l_1)$ of L_+^\uparrow , by the spin l ($l = |l_0|, |l_0| + 1, \dots, |l_1| - 1$), and by its projection m on the third axis, and $Q_{i'}^{\tau lm} \equiv Q_{i'}^{(l_0, l_1)lm}$ are matrix operators. The index i' in formula (4) can include some index i from relation (3) and, in addition, information about transformation properties of the corresponding bosonic field under the spatial reflection. The problem of the existence of a nontrivial doubly symmetric Lagrangian (4) and of matrix elements of the operator $Q_i^{(0,1)00}$ was solved in [2].

Sharing the multiply expressed belief [8] that monolocal infinite-component fields can effectively represent composite particles and using the results in [1] and [2], we study the possibility of describing hadrons and their interactions using the doubly symmetric theory of fields of the ISFIR class. For this possibility to have any chance of being realistic, the theory under consideration must satisfy at least the following two requirements. First, it must ensure the existence of physically acceptable mass spectra. Second, the elastic electromagnetic interaction of the ground fermionic state of the theory must be described by form factors similar to the nucleonic ones. The first requirement is addressed in this paper.

We recall that just the mass spectra were the main stumbling block in the previous attempts at a consistent relativistic description of particles with an infinite number of degrees of freedom, realized as different states of the same field, because in all the cases considered, the spectra had an accumulation point at zero. This conclusion regarding the mass spectra was drawn, in particular, by Ginzburg and Tamm [9], Yukawa [10], Shirokov [11], and Markov [12] in their analyses of certain particular bilocal equations and also by Gelfand and Yaglom [3] and Komar et al. [13] with regard to the general linear relativistically invariant equations based on representations of the group L_+^\uparrow that are decomposable into a finite direct sum of infinite-dimensional irreducible representations.

The result obtained in this work should therefore appear all the more important: the doubly symmetric theory of infinite-component fields of the ISFIR class has mass spectra that qualitatively reproduce the characteristic features of the experimental physics of hadrons and the parton bag model. We give its derivation and detailed formulation only for fermionic fields in the versions of the theory with one free parameter. Versions of the theory with three, five, and more free parameters with sufficiently high positions of the mass-spectrum levels (relative to the ground level) with any given precision reduce to the versions with a single parameter. Some of the mass-spectrum characteristics that we obtain occur as consequences of exact analytic calculations, and others are the result of applying numerical methods.

After solving the main problem, we give some attention in this paper to the problem of comparing the theoretical mass spectrum with the levels of hadron resonances. We then note a nontrivial feature of the ISFIR-class fields that a state of such a field can be manifested experimentally as a group of several nearby resonances with different spins. We indicate a possible identification of such groups in the case of nucleon resonances (see the table below). The confidence level of this identification cannot be established otherwise than by using an

extensive analysis involving, first, the expected correspondence between the theoretical and experimental nucleon and π -meson levels, second, the wave functions found for each of the theoretical states involved in the analysis, and, third, comparison of the results of theoretical (based on Lagrangian (4)) and experimental partial-wave analysis of the production amplitude of πN -states in the ranges of the observed nucleon resonances.

2. Equations and conditions for state vectors of particles described by ISFIR-class fields

The subject of our analysis is the linear relativistically invariant equations for a free fermionic field of the ISFIR class,

$$(\Gamma^\mu \partial_\mu + iR)\psi(x) = 0, \quad (5)$$

corresponding to Lagrangians (1). For a field corresponding to a plane wave with a zero spatial wave vector $\psi(x) = \exp(-iMt)\psi_{M0}$, Eq. (5) becomes

$$(M\Gamma^0 - R)\psi_{M0} = 0. \quad (6)$$

For each allowed value of the spin and its third projection, this equation becomes one or several recursive relations for the components of an infinite-component field ψ_{M0} . For any value of M , these relations themselves give one or several linearly independent solutions up to a numerical factor.

If Eqs. (6) were a closed mathematical problem, then it would most likely be complemented by the normalizability condition for its solutions. But in particle physics, we are interested not so much in the fields as in the amplitudes of the processes that are expressed through currents of the form $(\psi_{M0}, O\psi_{Mp})$ in the theory under consideration. The field vector ψ_{Mp} is obtained from the vector ψ_{M0} via a transition from one reference frame to another such that the wave four-vector $\{M, 0, 0, 0\}$ transforms into the four-vector $\{E, 0, 0, p\}$. The role of the operator O can be played by any of the matrix operators $Q_i^{\tau lm}$, Γ^μ , D^μ , or R in the theory. The currents $(\psi_{M0}, O\psi_{Mp})$ are in turn expressed through infinite series whose terms are quadratic in the components of the vectors ψ_{M0} and linear in the matrix elements of finite transformations of the proper Lorentz group. Therefore, instead of the normalizability condition for solutions of Eq. (6), we introduce a more restrictive condition that the relevant series converge, which we call the finite-amplitude condition. For any of the operators O above, the matrix elements relating the irreducible representations $\tau = (\pm 1/2, l_1)$ and $\tau' = (1/2, l_1 + n')$ or $\tau'' = (-1/2, l_1 + n'')$, where n' and n'' are integers, have the asymptotic behavior as $l_1 \rightarrow +\infty$ of the form l_1^β , where β is some constant. In view of this and of the asymptotic form of the field vector components ψ_{M0} and ψ_{Mp} as $l_1 \rightarrow +\infty$, which is given in this work, we can verify that for any fixed value of M , either all the series corresponding to the above currents converge or all of them diverge. We therefore formulate the finite-amplitude condition for any value of p as the relation

$$|(\psi_{M0}, R\psi_{Mp})| < +\infty \quad (7)$$

or

$$(\psi_{M'0}, R\psi_{Mp}) = a(p)\delta(M' - M), \quad (8)$$

where $a(p)$ is a nonzero number. Those and only those field vectors ψ_{M0} that satisfy Eq. (6) and relations (7) or (8) are called state vectors of a mass- M particle; this particle is a point in the respective discrete or continuum part of the mass spectrum.

Using formula (19) below and the appropriate analogue of formula (18), we can easily verify that if relation (8) is satisfied for some value $p = p_0$, then it cannot be satisfied for $p > p_0$.

Therefore, in the theory of ISFIR-class fields under consideration, the mass spectrum cannot have a continuum part.

For $p = 0$, finite-amplitude condition (7) or (8) becomes the normalizability condition for solutions of Eq. (6). We note that the finite-amplitude condition and the normalizability condition for solutions lead to the same mass spectra in all versions of the theory considered in Secs. 4–6 below. In some versions considered in Sec. 3, there is a nonempty discrete mass spectrum if the normalizability condition for solutions of Eq. (6) is satisfied, whereas the finite-amplitude requirement leads to an empty mass spectrum.

We now give the needed results in [1] and [2] concerning the doubly symmetric theory of ISFIR-class fields with the spontaneously broken secondary symmetry considered here.

The L_+^\uparrow -group representation S , under which the fields transforms, must coincide with one of the infinite-dimensional representations S^{k_1} (k_1 is a half-integer, $k_1 \geq 3/2$),

$$S^{k_1} = \sum_{n_1=0}^{+\infty} \sum_{n_0=-k_1+\frac{1}{2}}^{k_1-\frac{3}{2}} \oplus (\frac{1}{2} + n_0, k_1 + n_1). \quad (9)$$

In the representation space of S^{k_1} , the operator Γ^0 is given by

$$\begin{aligned} \Gamma^0 \xi_{(l_0, l_1)lm} = & c_0 D(l_1) \sqrt{\frac{(l-l_0)(l+l_0+1)(k_1-l_0-1)(k_1+l_0)}{(l_1-l_0)(l_1-l_0-1)(l_1+l_0)(l_1+l_0+1)}} \xi_{(l_0+1, l_1)lm} \\ & + c_0 D(l_1) \sqrt{\frac{(l-l_0+1)(l+l_0)(k_1-l_0)(k_1+l_0-1)}{(l_1-l_0+1)(l_1-l_0)(l_1+l_0-1)(l_1+l_0)}} \xi_{(l_0-1, l_1)lm} \\ & - c_0 D(l_0) \sqrt{\frac{(l_1-l)(l_1+l+1)(l_1-k_1+1)(l_1+k_1)}{(l_1-l_0)(l_1-l_0+1)(l_1+l_0)(l_1+l_0+1)}} \xi_{(l_0, l_1+1)lm} \\ & - c_0 D(l_0) \sqrt{\frac{(l_1-l-1)(l_1+l)(l_1-k_1)(l_1+k_1-1)}{(l_1-l_0-1)(l_1-l_0)(l_1+l_0-1)(l_1+l_0)}} \xi_{(l_0, l_1-1)lm}, \end{aligned} \quad (10)$$

where $\xi_{(l_0, l_1)lm}$ is a vector of the canonical basis in the space of the irreducible representation, $\tau = (l_0, l_1) \in S^{k_1}$, and c_0 is an arbitrary real constant. The function $D(j)$ of a half-integer argument j is given by the formula

$$D(j) = 1 \quad (11)$$

if the secondary symmetry of the theory is generated by a polar four-vector representation of L^\uparrow (Corollary 1 in [1]) and by the formula

$$D(j) = (-1)^{j-\frac{1}{2}} j \quad (12)$$

if the secondary symmetry of the theory is generated by an axial four-vector representation of L^\uparrow (Corollary 2 (A. 2) in [1]).

For the operator R in the representation space S^{k_1} and for both versions of the secondary symmetry in the considered theory, the same relations hold,

$$R \xi_{(l_0, l_1)lm} = r(l_0, l_1) \xi_{(l_0, l_1)lm} = \left[\sum_i \lambda^i q_i(l_0, l_1) \right] \xi_{(l_0, l_1)lm}, \quad (13)$$

$$q_i(-l_0, l_1) = q_i(l_0, l_1), \quad (14)$$

$$\begin{aligned} & (k_1 - l_0 - 1)(k_1 + l_0) q_i(l_0 + 1, l_1) + (k_1 - l_0)(k_1 + l_0 - 1) q_i(l_0 - 1, l_1) \\ & - (k_1 - l_1 - 1)(k_1 + l_1) q_i(l_0, l_1 + 1) - (k_1 - l_1)(k_1 + l_1 - 1) q_i(l_0, l_1 - 1) = \end{aligned}$$

$$= z_i(l_1 - l_0)(l_1 + l_0)q_i(l_0, l_1) \quad (15)$$

with $z_i = 2 - H_i^B/H^F$. The quantities H_i^B and H^F are normalization constants of the vector operators D^μ (with $D^\mu D_\mu = HE$) entering transformations (2) of the respective bosonic and fermionic fields $\varphi^i(x)$ and $\psi(x)$. They are independent of each other. For some fixed value of H^F , the quantity H_i^B and hence the parameter z_i can take any values. Indeed, if secondary symmetry transformation (2) of the bosonic field $\varphi^i(x)$ is nontrivial ($D^\mu \neq 0$) and this field is complex (just such fields were considered in [1] and [2]), then $H_i^B > 0$. If transformation (2) is nontrivial and the bosonic field is real, then $H_i^B < 0$. But if transformation (2) of the bosonic field is trivial, then $z_i = 2$, the operator $Q_i^{(0,1)00}$ is proportional to the unit operator E , and the existence of a nonzero vacuum expectation of this field does not affect the secondary symmetry.

In what follows, we only deal with fermionic fields transforming under the "lowest" of the proper Lorentz group representations S^{k_1} in (9), namely, the representation $S^{3/2}$, whose decomposition involves all finite-dimensional irreducible representations that contain spin $1/2$ and only such representations. A solution of recursive relation (15) for this representation ($k_1 = 3/2$) was expressed in [2] through the Gegenbauer polynomials and through hypergeometric series. It turns out that this solution can also be expressed through the elementary functions,

$$q_i\left(-\frac{1}{2}, l_1\right) = q_i\left(\frac{1}{2}, l_1\right) = 2q_{i0} \frac{u_i^N(u_i N + N + 1) - w_i^N(w_i N + N + 1)}{N(N+1)(u_i - w_i)(2 + u_i + w_i)}, \quad (16)$$

where $N = l_1 - 1/2$, $u_i = (z_i + \sqrt{z_i^2 - 4})/2$, and $w_i = (z_i - \sqrt{z_i^2 - 4})/2$.

In accordance with formulas (10) and (13), the operators Γ^0 and R are diagonal in the spin index l and in the spin projection index m , and their matrix elements are independent of m . Therefore, each vector ψ_{M0} satisfying Eq. (6) can be assigned certain values of spin and its projection. Components of this vector can be taken independent of the value of m . Linearly independent solutions of Eq. (6) can be chosen such that they have a definite P -parity. We recall that $P\xi_{(\pm 1/2, l_1)lm} = (-1)^{l-1/2}\xi_{(\mp 1/2, l_1)lm}$. If a vector $\sum_{l_1} [\chi(l_1)\xi_{(-1/2, l_1)lm} + \chi(l_1)\xi_{(1/2, l_1)lm}]$ with the P -parity $(-1)^{l-1/2}$ satisfies condition (7) and Eq. (6) with $M = M_0$, then the vector $\sum_{l_1} [-\chi(l_1)\xi_{(-1/2, l_1)lm} + \chi(l_1)\xi_{(1/2, l_1)lm}]$, which has the P -parity $(-1)^{l+1/2}$, also satisfies condition (7) and Eq. (6) but with $M = -M_0$.

In the representation space $S^{3/2}$ of the L_+^\dagger group, Eq. (6) for the field vector components $(\psi_{M0})_{(\pm 1/2, l_1)lm} \equiv \chi_{lm}(l_1)$ with the spatial parity $(-1)^{l-1/2}$ becomes

$$\begin{aligned} D\left(\frac{1}{2}\right) \frac{\sqrt{(l_1 - l)(l_1 + l + 1)}}{2l_1 + 1} \chi_{lm}(l_1 + 1) + D\left(\frac{1}{2}\right) \frac{\sqrt{(l_1 - l - 1)(l_1 + l)}}{2l_1 - 1} \chi_{lm}(l_1 - 1) \\ - \left[\frac{D(l_1)(2l + 1)}{4l_1^2 - 1} - \frac{1}{2Mc_0} r(l_1) \right] \chi_{lm}(l_1) = 0, \end{aligned} \quad (17)$$

where $l_1 \geq l$ and $r(l_1) \equiv r(\pm 1/2, l_1)$.

We write the relativistically invariant bilinear form of relation (7) via components of the vectors $\chi_{lm}(l_1)$ and via matrix elements of finite transformations of the proper Lorentz group,

$$(\psi_{M0}, R\psi_{Mp}) = \sum_{l_0 = -\frac{1}{2}}^{\frac{1}{2}} \sum_{l_1 = l+1}^{+\infty} (-1)^{l-\frac{1}{2}} \chi_{lm}^*(l_1) r(l_1) \{[\exp(\alpha I^{03})]_{(l_0, l_1)lm, (l_0, l_1)lm}\} \chi_{lm}(l_1), \quad (18)$$

where $\tanh \alpha = p/\sqrt{M^2 c^2 + p^2}$ and I^{03} is the infinitesimal operator of the group L_+^\dagger .

The explicit form of matrix elements of finite transformations of the L_+^\dagger group for infinite-dimensional unitary irreducible representations, denoted by a pair of numbers (l_0, ν) , was

found in [14]. The argument and the results in that work also hold for the finite-dimensional irreducible representations that interest us here. To preserve the precise meaning of the notation that we use, we must set $\nu = il_1$. In considering the convergence problem for the series of form (18), we need only know the asymptotic behavior of the relevant matrix elements as $l_1 \rightarrow +\infty$. We have

$$[\exp(\alpha I^{03})]_{(l_0, l_1)lm, (l_0, l_1)lm} = T_0 \frac{\exp(\alpha l_1)}{l_1} (1 + \mathcal{O}(l_1^{-1})), \quad (19)$$

where the quantity T_0 is independent of l_1 .

In what follows, we discuss the mass spectra in two versions of spontaneous secondary symmetry breaking: caused by one bosonic field of the ISFIR class (in which case the index i is to be omitted everywhere) or caused by two bosonic fields. In the first version, we separately consider three essentially different ranges of the z parameter values: $(-\infty, -2]$, $(-2, 2)$, and $(2, +\infty)$. In the second version, attention is given only to the range $z_1 \in (2, +\infty)$ and $z_2 \in (-2, 2)$.

3. Empty mass spectrum in the parameter range $z \in (-2, 2)$

In the case where the secondary symmetry of the theory is spontaneously broken, we cannot find solutions of Eq. (17) in the form of elementary or special functions, finite or infinite series. We also fail to find analytic formulas for mass spectra of the theory. But we can derive a number of conclusions regarding the mass spectra based on the asymptotic behavior of certain quantities.

Let $z \in (-2, 2)$. Using formula (16), we then have

$$r(l_1) = r_0 \frac{\sin \zeta l_1}{l_1} (1 + \mathcal{O}(l_1^{-1})) \quad (20)$$

as $l_1 \rightarrow +\infty$, where $\zeta \in (0, \pi)$ and r_0 is a constant. From this and Eq. (17), we obtain

$$\chi_{lm}(l_1) = A_0(-1)^{\lfloor \frac{l_1}{2} \rfloor} l_1^s (1 + K(l_1) + \mathcal{O}(l_1^{-2})) + B_0(-1)^{\lfloor \frac{l_1+1}{2} \rfloor} l_1^{-s} (1 - K(l_1) + \mathcal{O}(l_1^{-2})) \quad (21)$$

as $l_1 \rightarrow +\infty$, where

$$s = (2l + 1) \left(1 - D \left(\frac{1}{2} \right) \right), \quad K(l_1) = (-1)^{l_1 + \frac{1}{2}} \frac{r_0}{2M c_0 D \left(\frac{1}{2} \right) \sin \zeta} \cdot \frac{\cos \zeta l_1}{l_1}, \quad (22)$$

$[a]$ is the integer part of the number a , and the quantities A_0 and B_0 are independent of l_1 .

Based on this asymptotic formula, it is easy to establish that for $\alpha \neq 0$, the terms of series (18) grow as $l_1 \rightarrow +\infty$ independently of whether A_0 is equal to zero for certain values of M . Therefore, if $z \in (-2, 2)$, then a solution of Eq. (17) cannot satisfy finite-amplitude condition (7) at any value of M , and the mass spectrum is hence empty.

4. Characteristics of the mass spectra in the parameter range $z \in (-\infty, -2]$

Because the inequalities $w < -1$ and $-1 < u < 0$ hold in the range $z \in (-\infty, -2)$, it follows from (16) and (17) that as $l_1 \rightarrow +\infty$,

$$r(l_1) = r_0 \frac{w^{l_1 + \frac{1}{2}}}{l_1 + \frac{1}{2}} (1 + \mathcal{O}(l_1^{-1})), \quad (23)$$

$$\chi_{lm}(l_1) = A_0 G(l_1) (1 + \mathcal{O}(l_1^{-1})) + B_0 G^{-1}(l_1) (1 + \mathcal{O}(l_1^{-1})), \quad (24)$$

where

$$G(l_1) = \frac{v^{l_1 - \frac{1}{2}} w^{\frac{4l_1^2 - 1}{8}}}{(l_1 - \frac{1}{2})!}, \quad v = -\frac{r_0}{Mc_0 D(\frac{1}{2})}. \quad (25)$$

Obviously, if A_0 is nonzero for some values of M , then condition (7) cannot be satisfied. But if $A_0 = 0$ for some value of M , then the terms of series (18) have the asymptotic form of the order of $(l_1 + 1/2)!(l_1 - 1/2)!v^{-2l_1}u^{(4l_1^2 - 1)/4}\exp(\alpha l_1)$ as $l_1 \rightarrow +\infty$. The ratio of such a term of the series to the preceding term is equal to zero in the limit $l_1 \rightarrow +\infty$. Therefore, the relevant series (18) converges for the discussed value of M , which is equivalent to condition (7) being satisfied.

Therefore, for all values of the parameter z in the range $z \in (-\infty, -2)$ and for both versions of the theory expressed by relations (11) and (12), the mass spectrum is discrete whenever it is nonempty.

A similar statement also holds for $z = -2$. This is easy to verify taking into account that in this case, $r(l_1) = r_0(-1)^{l_1 + 1/2}l_1$ and an analogue of relation (24) holds with $G(l_1) = v^{l_1 - \frac{1}{2}}(-1)^{(4l_1^2 - 1)/8}(l_1 - 1/2)!$.

We now prove that in the range $z \in (-\infty, -2]$, the set of all masses of the theory is bounded from below by a positive number. For this, it suffices to find a number $\mu_0 > 0$ such that for the set of values of M satisfying the restriction $|M| \leq \mu_0$, the field components $\text{lm}(11)$ are not arbitrarily small for sufficiently large values of l_1 .

Using formula (16), we verify that in the range $z \leq -2$, the quantity $|r(l_1)|$ increases monotonically as l_1 increases and the function $r(l_1)$ of a half-integer argument has alternating signs, $r(l_1 + 1)/r(l_1) < 0$. Let $\mu_1 = |r(3/2)/c_0|$ and $|M| \leq \mu_1/2$. Then for both versions of the function $D(j)$ (Eqs. (11) and (12)) and for all the allowed values of l and l_1 , the quantity $D(l_1)(l + 1/2)/(l_1^2 - 1/4) - r(l_1)/Mc_0$ is greater than one in absolute value and changes its sign as l_1 changes by 1. Together with Eq. (17), this gives the desired inequality $|\chi_{lm}(l_1 + 1)| > |\chi_{lm}(l_1)|$.

For convenience in what follows, we use one or another relation between the normalization constants, thus obtaining different mass units. If $r(3/2)/(2c_0 D(1/2)) = \pm 1$ (the plus and minus signs refer to the respective relations (11) and (12), chosen such that the lower levels of the spin-1/2 particle have the spatial parity +1), we introduce the notation $M_c = |M|$. Any number of the lower values of mass M_c can be found using numerical methods. In accordance with the above, only the range $M_c > 0.5$ is to be considered in numerical calculations. To find all points of the spectrum in the relevant ranges of M_c and of the parameter z for a fixed spin l , it suffices to restrict to seeking points M_c at which the quantity A_0 in Eq. (24) vanishes but does not have a minimum or a maximum for some fixed value of z . In arbitrary small neighborhoods of such points in the mass spectrum, the quantity $\chi_{lm}(l_1)$ then obviously changes its sign for sufficiently large values of l_1 . This plays the role of an algorithm for solving the mass problem numerically. Analyzing the dependence of any two chosen neighboring points of the mass spectrum on the parameter z , we can find whether a value z_0 exists such that in tending to it from one side, these points become arbitrarily close to each other but do not appear on the other side of z_0 . If such a number z_0 exists, then the limit value of the two chosen points is the mass value M_c at which the quantity A_0 has the zero value and an extremum.

In Figs. 1 and 2, in the cases corresponding to the respective relations (11) and (12), we show the dependence of the masses of the states with spins 1/2, 3/2, 5/2, and 7/2 on the parameter z .

We first note several characteristics of the mass spectrum of the theory with double symmetry generated by the axial four-vector representation of the orthochronous Lorentz group (which corresponds to relation (12)). First, the mass spectrum is nonempty if the spatial parity of spin- l particles is $(-1)^{l-1/2}$ and is empty if the parity is $(-1)^{l+1/2}$. Second, among the mass

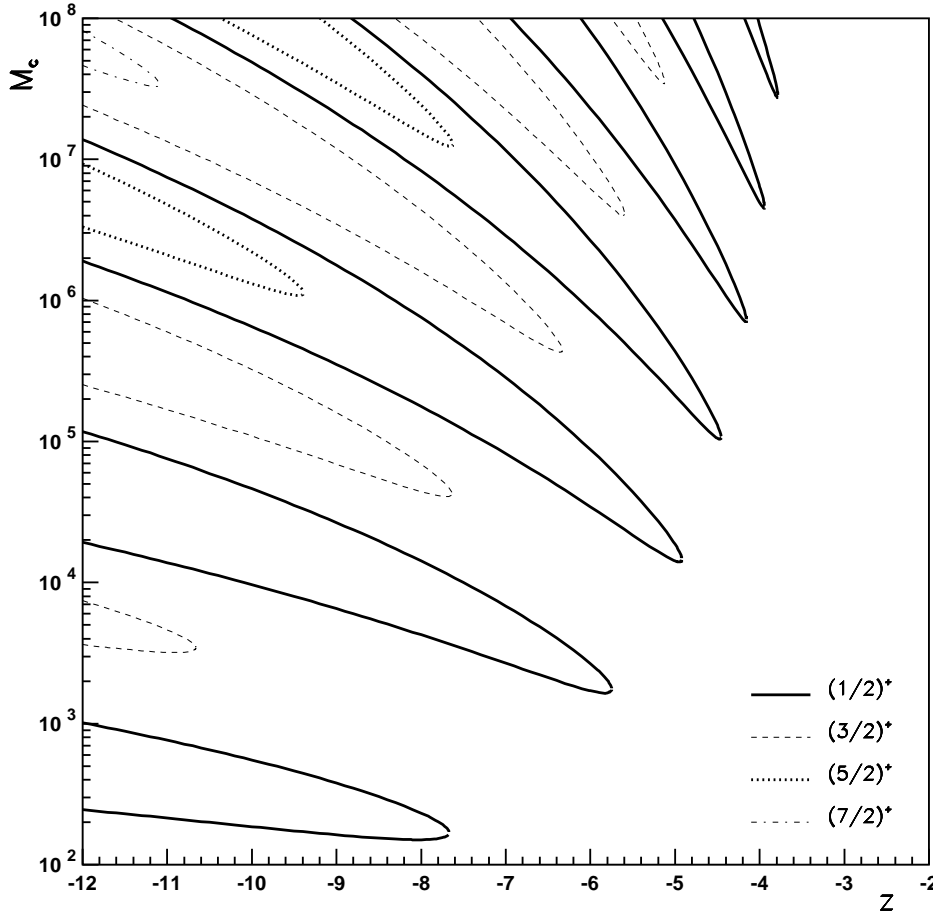


Figure 1: Dependence of the mass levels on the parameter z for $z < -2$ in the theory with double symmetry generated by the polar four-vector representation of the orthochronous Lorentz group.

lines, there are pairs that terminate, merging at certain common limit points for $z_0 < -2$; there are also single lines that exist in the entire range $z \in (-\infty, -2]$. Third, for all z , from the range $(-\infty, -2.1)$ at least, the levels of the mass spectrum have the same ordering in accordance with spin. It is the same as for $z = -2.645$ for example, where the lower masses M_c (with spin and parity l^P) are given by $1.982(1/2)^+$, $3.209(3/2)^-$, $3.687(1/2)^+$, $5.470(5/2)^+$, $6.143(3/2)^-$, $6.964(1/2)^+$, $9.709(7/2)^-$, $10.72(5/2)^+$, $11.91(3/2)^-$, $13.35(1/2)^+$, $17.82(9/2)^+$, and $19.38(7/2)^-$. The lowest-level masses with a given spin increase as the spin increases somewhat faster than the geometric progression. A sequence of mass levels taken in consecutive order with the same spin is close to the geometric progression. Therefore, although the ratio of lowest-level masses with the spin and parity $(3/2)^-$ and $(1/2)^+$ in the above example is equal to the mass ratio of the $N(1520)$ resonance and the nucleon, the positions of levels with $l^P = (5/2)^+$, $(7/2)^-$, $(9/2)^+$, etc., are drastically different from the positions of the corresponding nucleonic resonances [15]. Fourth, for the parameter values close to $z = -2$, the level ordering in the mass spectrum in accordance with spin changes as z changes. For example, for $z = -2$, the lower levels M_c (l^P) are $9.506(5/2)^+$, $12.14(9/2)^+$, $12.55(3/2)^-$, $13.25(5/2)^+$, $15.77(13/2)^+$, $16.03(19/2)^-$, and $17.45(21/2)^+$.

Among the mass-spectrum characteristics in the theory with double symmetry generated by the polar four-vector representation of the L^\uparrow group (which corresponds to relation (11)), we note the following. First, particles at any value of spin have the same spatial parity $+1$. Second, the set of all mass lines is decomposed into pairs that merge and terminate at $z_0 < -2$. Third, the lower-level masses with two consecutive spins, whenever they exist for a given z in

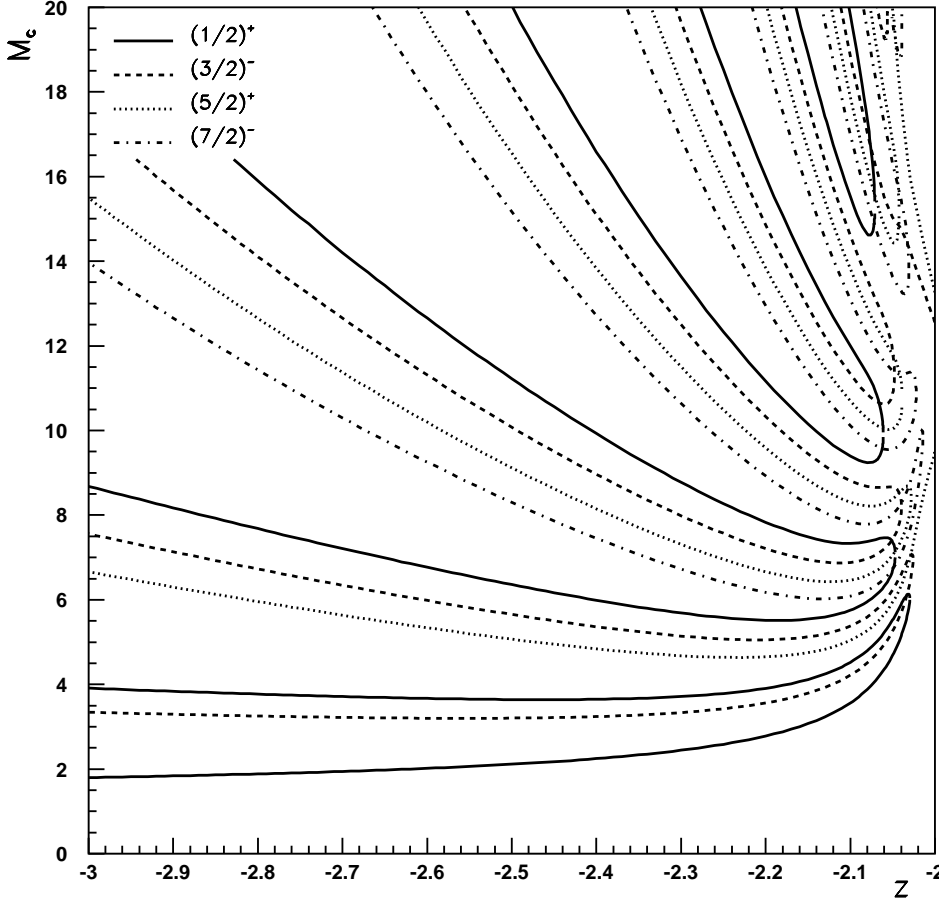


Figure 2: Dependence of the mass levels on the parameter z for $z < -2$ in the theory with double symmetry generated by the axial four-vector representation of the orthochronous Lorentz group.

the range under consideration, differ from each other by an order of magnitude at least, which manifestly contradicts the baryon resonance picture.

5. Characteristics of the mass spectra in the parameter range $z \in (2, +\infty)$

Because the inequalities $u > 1$ and $0 < w < 1$ are satisfied in the range $z \in (2, +\infty)$, formulas (23)–(25) hold, and the subsequent argument regarding the validity of condition (7) is applicable if w is replaced with u and u with w in these formulas and in the corresponding argument. This implies the conclusion that for all values of $z \in (2, +\infty)$ in both versions of the theory corresponding to relations (11) and (12), the mass spectrum is discrete if it is nonempty.

It follows from relation (16) that in the range $z > 2$, the quantity $r(l_1)/r(3/2)$ is positive and increases monotonically as l_1 increases. This fact and Eq. (17) with $l_1 > l$ lead to the inequality $|\chi_{lm}(l_1)| > (1 + 1/(l_1 - l))|\chi_{lm}(l)|$ for all l and for all values of M in the range $|M| \leq \mu_1/3$, where $\mu_1 = |r(3/2)/c_0|$. In the range of M specified, therefore, series (18) diverges, condition (7) is not satisfied, and mass-spectrum points are absent, i.e., the mass spectrum is bounded from below.

In Figs. 3 and 4, corresponding to the respective formulas (11) and (12), we show the dependence of the lower mass levels on the parameter z for spins $1/2$, $3/2$, $5/2$, and $7/2$, each of which is assigned the spatial parities $+1$ and -1 . As $z \rightarrow 2+0$, the masses M_c tend to unity on all lines in both versions of the theory. For any $z > 2$, the masses of several lower levels with consecutive spins l and with the parities $(-1)^{l-1/2}$ approximate a geometric progression. This

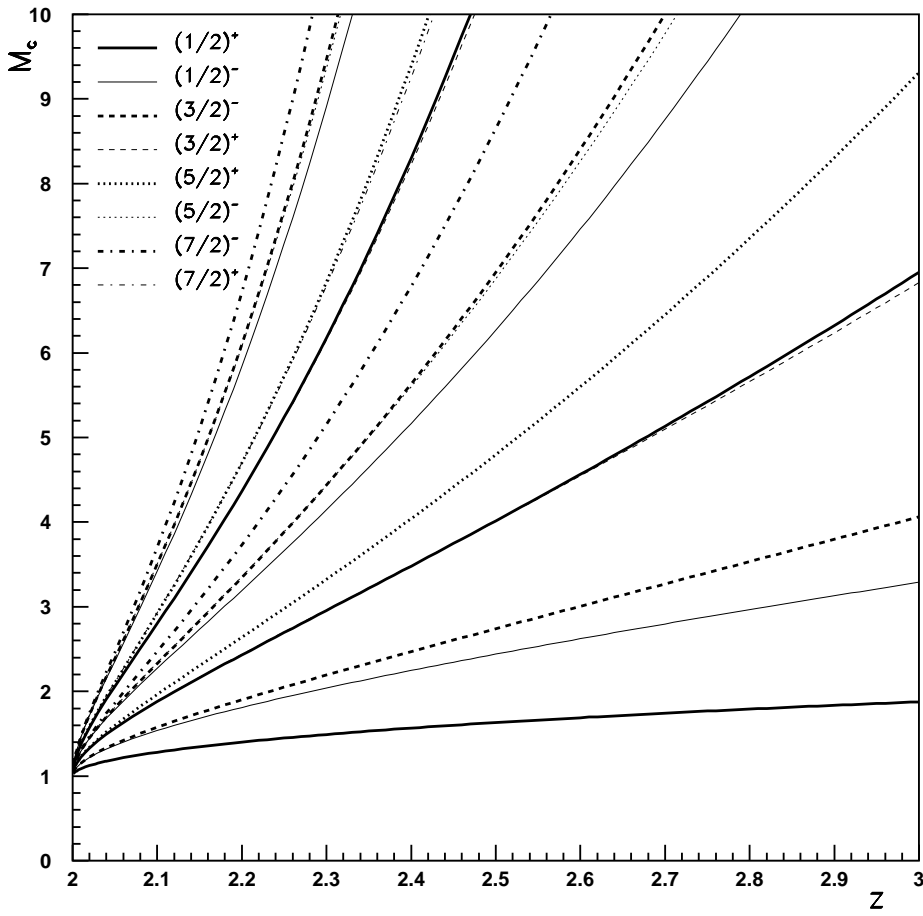


Figure 3: Dependence of the mass levels on the parameter z for $z > 2$ in the theory with double symmetry generated by the polar four-vector representation of the orthochronous Lorentz group.

property also applies to the masses of several consecutive levels with the same spin and parity. This does not allow achieving any reasonable quantitative agreement between the lower levels of the theory with one parameter z and the levels of nucleonic resonances. An essential difference between the two versions of the theory is in the order of relative positions of the lower levels with two consecutive spin values and a given parity. To illustrate all this, we give numerical examples below, where the mass ratio of the lower levels with $l^P = (3/2)^-$ and $l^P = (1/2)^+$ in the theory considered is the same as for the $N(1520)$ resonance and the nucleon.

We note several characteristic features of the mass spectrum for the version of the theory with relation (11). First, the mass-level ordering in accordance with spin and parity is the same for all values of the parameter z from the range $(2, +\infty)$. It is such as for $z = 2.441$ for example, when the lowest masses M_c and the corresponding spin and parity l^P are given by $1.595(1/2)^+$, $2.328(1/2)^-$, $2.582(3/2)^-$, $3.699(1/2)^+$, $3.700(3/2)^+$, $4.346(5/2)^+$, $5.609(1/2)^-$, $6.111(5/2)^-$, $6.158(3/2)^-$, $7.522(7/2)^-$, $9.272(1/2)^+$, and $9.174(3/2)^+$. Second, certain mass levels with different l^P differ from each other insignificantly in a considerable neighborhood of the point $z = 2$. Such a small difference occurs for the $(n+1)$ th level with $l^P = (1/2)^+$ and the n th level with $l^P = (3/2)^+$, for the $(n+1)$ th level with $l^P = (3/2)^-$ and the n th level with $l^P = (5/2)^-$, and for the $(n+1)$ th level with $l^P = (5/2)^+$ and the n th level with $l^P = (7/2)^+$ ($n \geq 1$), etc.

For the version of the theory with relation (12), the general level ordering in accordance with spin and parity can be different for different values of the parameter z . But the ordering of the lowest levels with some l^P is the same for all $z > 2$. This ordering is as in the following example,

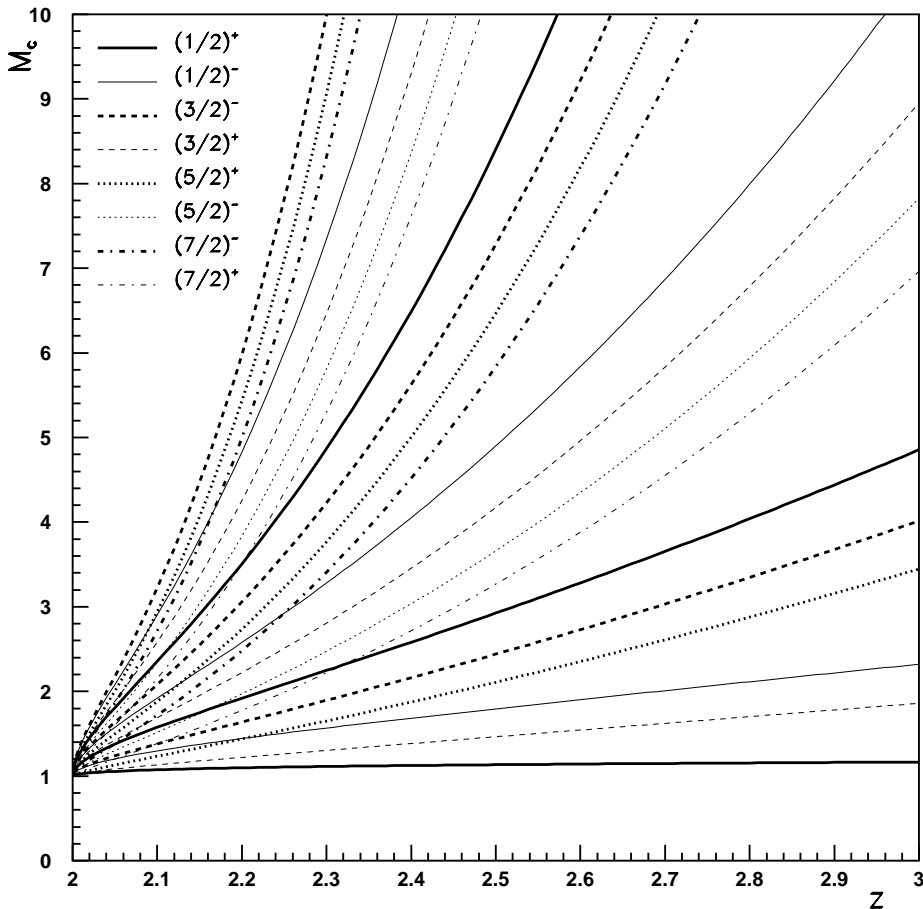


Figure 4: Dependence of the mass levels on the parameter z for $z > 2$ in the theory with double symmetry generated by the axial four-vector representation of the orthochronous Lorentz group.

where all the levels with $M_c < 4$ are given, evaluated at $z = 2.261$: $1.108(1/2)^+$, $1.272(3/2)^+$, $1.521(1/2)^-$, $1.567(5/2)^+$, $1.794(3/2)^-$, $2.041(7/2)^+$, $2.119(1/2)^+$, $2.272(5/2)^-$, $2.573(3/2)^+$, $2.769(9/2)^+$, $2.999(1/2)^-$, $3.027(7/2)^-$, $3.342(5/2)^+$, $3.747(3/2)^-$, and $3.876(11/2)^+$.

On the whole, therefore, the doubly symmetric theory with one parameter z connected with spontaneous secondary symmetry breaking gives the following qualitative picture of the mass spectra. The continuum part of the mass spectrum does not exist for any values of z . If the mass spectrum is nonempty for some z , then the mass levels are bounded from below by a positive quantity. In the parameter range $z > 2$, each value of spin and parity corresponds to a countable set of masses extending up to infinity, and the lowest mass level values with a given spin increase as the spin increases. The theory with one parameter z does not give a satisfactory quantitative agreement with the nucleon resonance levels.

6. Comparison of the mass spectrum of the theory with two parameters z_i with nucleon resonance levels

The general situation with the mass spectra of the theory involving two bosonic fields of the ISFIR class with nonvanishing vacuum expectations of their scalar components (scalar with respect to the group L^\uparrow) is quite rich with different versions. In what follows, we consider only one version in some detail with the aim of a tentative quantitative comparison of the theoretical mass spectrum with the nucleon resonance levels.

We consider the theory with the double symmetry generated by the polar four-vector representation of the orthochronous Lorentz group (with the corresponding relation (11)). With

two parameters z_i , the quantity $r(l_1)$ in Eq. (17) can be written as

$$r(l_1) = f_0 \left(\frac{q_1(\frac{1}{2}, l_1)}{q_{10}} + f_2 \frac{q_2(\frac{1}{2}, l_1)}{q_{20}} \right), \quad (26)$$

where $q_i(1/2, l_1)$ depends on z_i and is given by formula (16). It is obvious that in addition to the normalization constant f_0/c_0 , this theory involves three free parameters: z_1 , z_2 , and f_2 . We choose the following restrictions on them: $z_1 > 2$, $|z_2| < 2$, and $|f_2| < 1$. In this parameter range, as l_1 increases, the quantity $|q_1(1/2, l_1)|$ increases monotonically and $q_2(1/2, l_1)$ oscillates with a decreasing amplitude. The quantities $r(l_1)$ and $f_0 q_1(1/2, l_1)/q_{10}$ have the same sign for all l_1 and can be significantly different from each other only for small values of l_1 . The parameter z_2 has no effect on the asymptotic behavior of the quantities $r(l_1)$ and $\chi_{lm}(l_1)$ as $l_1 \rightarrow +\infty$. Formulas (23)–(25) are valid if w in them is replaced with u_1 ; the conclusions regarding the mass spectra following from these formulas are also valid. Therefore, in the considered version of the theory with two parameters z_1 and z_2 , compared with the version of the theory with a single parameter z_1 , only mass spectrum levels with the lowest values of the spin l can significantly change their positions.

We now note a very important circumstance inherent in the theory of infinite-component fields considered. Let the ground fermionic and the ground bosonic levels of the theory correspond to the respective particles F with spin $1/2$ and B with spin 0 , and let an excited fermionic level exist in the theory with the corresponding particle F^* of spin l . We consider the amplitude \mathcal{M} of the decay $F^* \rightarrow FB$ in the rest frame of the resonance F^* described by Lagrangian (4). Because the particles F and B have nonzero velocities in this reference frame, the corresponding infinite-component fields of the ISFIR class have nonzero components with the respective all half-integer and all integer spins. Therefore, the amplitude \mathcal{M} has nonzero terms in which the components of the F^* , F , and B fields are described by the following sets of spins $\{l_{F^*}, l_F, l_B\}$: $\{l, l, 0\}$, $\{l, l-1, 1\}$, $\{l, l, 1\}$, $\{l, l+1, 1\}$, $\{l, l-2, 2\}$, \dots , $\{l, l+2, 2\}$, \dots . The contribution of a given spin l' of the fermion F and spin l'' of the boson B to the amplitude \mathcal{M} can be estimated only if the state vectors of the particles F^* , F , and B are known in their rest frames. Because experimental conclusions about a resonance spin are obtained from the partial-wave analysis, based on the representations of the three-dimensional rotation group but not on representations of the Lorentz group, it follows that the aforesaid may be manifested in the experimentalist opinion regarding the existence of a group of several resonances that have the same mass and differ from each other only by their spins. Therefore, in comparing the theoretical mass spectrum with the experimental one, we must pay special attention to such groups of resonances and make the decision regarding the number of the corresponding theoretical levels.

The simplest proposed correspondence between the theory version under consideration and the experimental picture of nucleon resonances is given in Table 1. It corresponds to the normalization constant and the free parameters chosen as $f_0/c_0 = -939/2.4686$ MeV, $z_1 = 2.036$, $z_2 = 0.14$, and $f_2 = -0.6724$. The parameters z_2 and f_2 are chosen such that the lowest-level masses with $l^P = (3/2)^-$ and $l^P = (5/2)^+$ are respectively equal to 1508 and 1675 MeV. The parameter $z_1 = 2.036$ determines particle states with large spins. As the experimental masses of almost all resonances, we take their pole positions. The Breit–Wigner masses (indicated with the BW superscript in the table) are taken, first, for those few resonances whose pole positions are not given in [15] and, second, for the Roper resonance $N(1440)$ because two poles, $1370 - 114i$ and $1360 - 120i$, have been found in the neighborhood of 1440 MeV with their parameters strongly different from the usual $M = 1470$ MeV and $\Gamma = 545$ MeV [16].

A tentative comparison of the mass spectra of the theory considered here with levels of nonnucleon resonances has not been made yet. In particular, it requires the singlet–octet

separation of the Λ resonances and an octet–decuplet separation of the Σ and Ξ resonances in accordance with the internal symmetry group $SU(3)$. The description of the Δ resonances is supposedly to be given by the $S^{5/2}$ representation of the L_+^\uparrow group described by formula (11) with $k_1 = 5/2$. In this case, the lowest mass level has the spin $3/2$. Analyzing mass spectra of bosonic fields of the ISFIR class in any versions of the doubly symmetric theory will become possible only after finding the structure of the Lagrangian of the four-particle interaction of such fields with each other and the ensuing spontaneous violation of secondary symmetry.

Table 1

Theory		Experiments [15]			
l^P	Mass (MeV)	l^P	Resonance	Mass (MeV)	Status
$\frac{1}{2}^+$	939	$\frac{1}{2}^+$	N	939	****
$\frac{1}{2}^+$	1481	$\frac{1}{2}^+$	$N(1440)$	1430-1470 ^{BW}	****
$\frac{1}{2}^-$	1487	$\frac{1}{2}^-$	$N(1535)$	1495-1515	****
$\frac{3}{2}^-$	1508	$\frac{3}{2}^-$	$N(1520)$	1505-1515	****
$\frac{3}{2}^+$	1661	$\frac{1}{2}^-$	$N(1650)$	1640-1680	****
		$\frac{5}{2}^-$	$N(1675)$	1655-1665	****
		$\frac{3}{2}^+$	$N(1720)$	1650-1750	****
$\frac{5}{2}^+$	1675	$\frac{5}{2}^+$	$N(1680)$	1665-1675	****
		$\frac{3}{2}^-$	$N(1700)$	1630-1730	***
		$\frac{1}{2}^+$	$N(1710)$	1670-1770	***
$\frac{5}{2}^-$	1892	$\frac{3}{2}^+$	$N(1900)$	$\approx 1900^{\text{BW}}$	**
		$\frac{7}{2}^+$	$N(1990)$	1870-1930	**
$\frac{7}{2}^-$	1923	$\frac{7}{2}^-$	$N(2190)$	1950-2150	****
$\frac{3}{2}^-$	1940	$\frac{5}{2}^+$	$N(2000)$	$\approx 2000^{\text{BW}}$	**
$\frac{1}{2}^+$	1995	$\frac{3}{2}^-$	$N(2080)$	1980-2120	**
$\frac{5}{2}^+$	2004	$\frac{1}{2}^+$	$N(2100)$	2080-2160	*
$\frac{3}{2}^+$	2144	$\frac{1}{2}^-$	$N(2090)$	2080-2220	*
$\frac{7}{2}^+$	2140	$\frac{5}{2}^-$	$N(2200)$	2040-2160	**
$\frac{9}{2}^+$	2179	$\frac{9}{2}^-$	$N(2250)$	2080-2200	****
$\frac{11}{2}^-$	2244	$\frac{9}{2}^+$	$N(2220)$	2100-2240	****
$\frac{13}{2}^+$	2547	$\frac{11}{2}^-$	$N(2600)$	2550-2750 ^{BW}	***
$\frac{15}{2}^+$	2919	$\frac{13}{2}^+$	$N(2700)$	$\approx 2700^{\text{BW}}$	*

Acknowledgments. The author is very grateful to E.E.Boos, V.I.Savrin, and N.P.Yudin for the useful discussions and the support of his work.

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